Integration

PPLATO PROMOTING PHYSICS LEARNING & TEACHING OPPORTUNITIES

SUBSTITUTION II .. $\frac{1}{f(x)} \cdot f'(x)$

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A Tutorial Module for practising the integration of expressions of the form $\frac{1}{f(x)} \cdot f'(x)$

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1. Theory

Consider an integral of the form

$$\int \frac{f'(x)}{f(x)}\,dx$$

Letting $u = f(x)$ then $\dfrac{du}{dx} = f'(x)$, and this gives $du = f'(x)dx$

So

$$\begin{aligned}\int \frac{f'(x)}{f(x)}\,dx &= \int \frac{1}{f(x)} \cdot f'(x)\,dx \\ &= \int \frac{du}{u} \\ &= \ln|u| + C\end{aligned}$$

The general result is

$$\boxed{\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C}$$

This is a useful generalisation of the standard integral

$$\int \frac{1}{x}\,dx = \ln|x| + C\,,$$

that applies straightforwardly when the top line of what we are integrating is a multiple of the derivative of the bottom line

2. Exercises

Click on EXERCISE links for full worked solutions (there are 11 exercises in total).

Perform the following integrations:

Exercise 1.

$$\int \frac{10x+3}{5x^2+3x-1}\,dx$$

Exercise 2.

$$\int \frac{3x^2}{x^3-6}\,dx$$

Exercise 3.

$$\int \frac{4x^2}{x^3-1}\,dx$$

Exercise 4.

$$\int \frac{2x-1}{x^2-x-7}\,dx$$

Exercise 5.

$$\int \frac{x-1}{x^2-2x+7}\,dx$$

Exercise 6.

$$\int \frac{8x}{x^2-4}\,dx$$

Exercise 7.

$$\int \frac{x^2}{x^3+2}\,dx$$

Exercise 8.

$$\int \frac{\sin x}{\cos x}\, dx$$

Exercise 9.

$$\int \frac{\cos x}{\sin x}\, dx$$

Exercise 10.

$$\int \frac{\cos x}{5+\sin x}\, dx$$

Exercise 11.

$$\int \frac{\sinh x}{1-\cosh x}\, dx \ ,$$

3. Answers

1. $\ln|5x^2 + 3x - 1| + C$,
2. $\ln|x^3 - 6| + C$,
3. $\frac{4}{3}\ln|x^3 - 1| + C$,
4. $\ln|x^2 - x - 7| + C$,
5. $\frac{1}{2}\ln|x^2 - 2x + 7| + C$,
6. $4\ln|x^2 - 4| + C$,
7. $\frac{1}{3}\ln|x^3 + 2| + C$
8. $-\ln|\cos x| + C$,
9. $\ln|\sin x| + C$,
10. $\ln|5 + \sin x| + C$,
11. $-\ln|1 - \cosh x| + C$.

4. Standard integrals

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
x^n	$\frac{x^{n+1}}{n+1} \quad (n \neq -1)$	$[g(x)]^n g'(x)$	$\frac{[g(x)]^{n+1}}{n+1} \quad (n \neq -1)$
$\frac{1}{x}$	$\ln\|x\|$	$\frac{g'(x)}{g(x)}$	$\ln\|g(x)\|$
e^x	e^x	a^x	$\frac{a^x}{\ln a} \quad (a > 0)$
$\sin x$	$-\cos x$	$\sinh x$	$\cosh x$
$\cos x$	$\sin x$	$\cosh x$	$\sinh x$
$\tan x$	$-\ln\|\cos x\|$	$\tanh x$	$\ln\cosh x$
$\operatorname{cosec} x$	$\ln\left\|\tan\frac{x}{2}\right\|$	$\operatorname{cosech} x$	$\ln\left\|\tanh\frac{x}{2}\right\|$
$\sec x$	$\ln\|\sec x + \tan x\|$	$\operatorname{sech} x$	$2\tan^{-1} e^x$
$\sec^2 x$	$\tan x$	$\operatorname{sech}^2 x$	$\tanh x$
$\cot x$	$\ln\|\sin x\|$	$\coth x$	$\ln\|\sinh x\|$
$\sin^2 x$	$\frac{x}{2} - \frac{\sin 2x}{4}$	$\sinh^2 x$	$\frac{\sinh 2x}{4} - \frac{x}{2}$
$\cos^2 x$	$\frac{x}{2} + \frac{\sin 2x}{4}$	$\cosh^2 x$	$\frac{\sinh 2x}{4} + \frac{x}{2}$

$f(x)$	$\int f(x)\,dx$	$f(x)$	$\int f(x)\,dx$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$	$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert \quad (0<\lvert x\rvert<a)$
	$(a>0)$	$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert \quad (\lvert x\rvert>a>0)$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a}$	$\frac{1}{\sqrt{a^2+x^2}}$	$\ln\left\lvert\frac{x+\sqrt{a^2+x^2}}{a}\right\rvert \quad (a>0)$
	$(-a<x<a)$	$\frac{1}{\sqrt{x^2-a^2}}$	$\ln\left\lvert\frac{x+\sqrt{x^2-a^2}}{a}\right\rvert \quad (x>a>0)$
$\sqrt{a^2-x^2}$	$\frac{a^2}{2}\left[\sin^{-1}\left(\frac{x}{a}\right)\right.$	$\sqrt{a^2+x^2}$	$\frac{a^2}{2}\left[\sinh^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{a^2+x^2}}{a^2}\right]$
	$\left.+\frac{x\sqrt{a^2-x^2}}{a^2}\right]$	$\sqrt{x^2-a^2}$	$\frac{a^2}{2}\left[-\cosh^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{x^2-a^2}}{a^2}\right]$

5. Tips

- STANDARD INTEGRALS are provided. Do not forget to use these tables when you need to

- When looking at the THEORY, STANDARD INTEGRALS, ANSWERS or TIPS pages, use the Back button (at the bottom of the page) to return to the exercises

- Use the solutions intelligently. For example, they can help you get started on an exercise, or they can allow you to check whether your intermediate results are correct

- Try to make less use of the full solutions as you work your way through the Tutorial

Full worked solutions

Exercise 1.

$$\int \frac{10x+3}{5x^2+3x-1}\,dx \text{ is of the form } \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C.$$

Let $u = 5x^2 + 3x - 1$ then $\frac{du}{dx} = 10x + 3$, and $du = (10x + 3)dx$

$$\begin{aligned} \therefore \int \frac{10x+3}{5x^2+3x-1}\,dx &= \int \frac{du}{u} \\ &= \ln|u| + C \\ &= \ln|5x^2+3x-1| + C\,. \end{aligned}$$

Return to Exercise 1

Exercise 2.

$$\int \frac{3x^2}{x^3-6}\,dx \text{ is of the form } \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C.$$

Let $u = x^3 - 6$ then $\frac{du}{dx} = 3x^2$ and $du = 3x^2 dx$

$$\begin{aligned} \therefore \int \frac{3x^2}{x^3-6}\,dx &= \int \frac{1}{u}\,du \\ &= \ln|u| + C \\ &= \ln|x^3-6| + C\,. \end{aligned}$$

Return to Exercise 2

Exercise 3.

$$\int \frac{4x^2}{x^3-1}\,dx \quad \text{is of the slightly more general form}$$

$$\int k\frac{f'(x)}{f(x)}\,dx = k\int \frac{f'(x)}{f(x)}\,dx = k\ln|f(x)| + C,$$

where k is a constant
i.e. the top line is a **multiple of the derivative** of the bottom line.

Let $u = x^3 - 1$ then $\frac{du}{dx} = 3x^2$ and $\frac{du}{3} = x^2dx$

$$\begin{aligned} \therefore\ 4\int \frac{x^2}{x^3-1}dx &= 4\int \frac{1}{u}\frac{du}{3} \\ &= \frac{4}{3}\ln|u| + C \\ &= \frac{4}{3}\ln|x^3-1| + C\,. \end{aligned}$$

Return to Exercise 3

Exercise 4.

$$\int \frac{2x-1}{x^2-x-7}\,dx \text{ is of the form } \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C.$$

Let $u = x^2 - x - 7$ then $\dfrac{du}{dx} = 2x - 1$ and $du = (2x-1)dx$

$$\begin{aligned}\therefore \int \frac{2x-1}{x^2-x-7}\,dx &= \int \frac{du}{u} \\ &= \ln|u| + C \\ &= \ln|x^2-x-7| + C\,.\end{aligned}$$

Return to Exercise 4

Exercise 5.

$$\int \frac{x-1}{x^2-2x+7}\,dx \text{ is of the slightly more general form}$$

$$\int k\frac{f'(x)}{f(x)}\,dx = k\int \frac{f'(x)}{f(x)}\,dx = k\ln|f(x)| + C,$$

where k is a constant

i.e. the top line is a **multiple of the derivative** of the bottom line.

Let $u = x^2 - 2x + 7$ then $\frac{du}{dx} = 2x - 2$ and $du = (2x-2)dx$

so that $\frac{du}{2} = (x-1)dx$

$$
\begin{aligned}
\therefore \int \frac{x-1}{x^2-2x+7}\,dx = \int \frac{1}{u}\frac{du}{2} &= \frac{1}{2}\int \frac{du}{u} \\
&= \frac{1}{2}\ln|u| + C \\
&= \frac{1}{2}\ln|x^2-2x+7| + C\,.
\end{aligned}
$$

Return to Exercise 5

Exercise 6.

$$\int \frac{8x}{x^2-4}\,dx$$

Let $u = x^2 - 4$ then $du = 2x\,dx$ and $\dfrac{du}{2} = xdx$

$$\begin{aligned} \therefore \int \frac{8x}{x^2-4}dx &= 8\int \frac{1}{u}\frac{du}{2} \\ &= 4\ln|u| + C \\ &= 4\ln|x^2-4| + C\,. \end{aligned}$$

Return to Exercise 6

Exercise 7.

$$\int \frac{x^2}{x^3+2}\,dx$$

Let $u = x^3 + 2$ then $du = 3x^2dx$ and $\dfrac{du}{3} = x^2dx$

$$\begin{aligned} \therefore \int \frac{x^2}{x^3+2}dx &= \int \frac{1}{u}\frac{du}{3} \\ &= \frac{1}{3}\ln|u| + C \\ &= \frac{1}{3}\ln|x^3+2| + C\,. \end{aligned}$$

Return to Exercise 7

Exercise 8.

$\int \frac{\sin x}{\cos x}\, dx$

Let $u = \cos x$ then $\frac{du}{dx} = -\sin x$ and $\frac{du}{(-1)} = \sin x\, dx$

$$\begin{aligned} \therefore \int \frac{\sin x}{\cos x}\, dx &= \int \frac{1}{u} \frac{du}{(-1)} \\ &= -\int \frac{du}{u} \\ &= -\ln|u| + C \\ &= -\ln|\cos x| + C. \end{aligned}$$

Return to Exercise 8

Exercise 9.

$\int \frac{\cos x}{\sin x}\, dx$

Let $u = \sin x$ then $\frac{du}{dx} = \cos x$ and $du = \cos x\, dx$

$$
\begin{aligned}
\therefore \int \frac{\cos x}{\sin x}\, dx &= \int \frac{1}{u}\, du \\
&= \ln |u| + C \\
&= \ln |\sin x| + C.
\end{aligned}
$$

Return to Exercise 9

Exercise 10.

$$\int \frac{\cos x}{5+\sin x}\,dx$$

Let $u = 5 + \sin x$ then $\dfrac{du}{dx} = \cos x$ and $du = \cos x\,dx$

$$\begin{aligned} \therefore \int \frac{\cos x}{5+\sin x}\,dx &= \int \frac{du}{u} \\ &= \ln|u| + C \\ &= \ln|5+\sin x| + C\,. \end{aligned}$$

Return to Exercise 10

Exercise 11.

$$\int \frac{\sinh x}{1-\cosh x}\, dx$$

Let $u = 1-\cosh x$ then $\dfrac{du}{dx} = -\sinh x$ and $-du = \sinh x\, dx$

$$\begin{aligned} \therefore \int \frac{\sinh x}{1-\cosh x}\, dx &= \int \frac{1}{u} \cdot (-du) \\ &= -\ln|u| + C \\ &= -\ln|1-\cosh x| + C\,. \end{aligned}$$

Return to Exercise 11